A study on nonsymmetric cone normed spaces

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Abstract

In the realms of theoretical computer science and approximation theory, asymmetric normed spaces play an important role. In this paper, by combining asymmetric norm and cone norm, it is defined asymmetric cone normed spaces. Also, it is introduced and studied some topological concepts in asymmetric cone normed spaces.

Keywords: Asymmetric norm, cone norm, convergence, completeness.

Simetrik olmayan konik normlu uzaylar üzerine bir çalışma

Öz

Teorik bilgisayar bilimi ve yaklaşım teorisi alanlarında asimetrik normlu uzaylar önemli bir rol oynamaktadır. Bu çalışmada asimetrik norm ve konik norm birleştirilerek asimetrik konik normlu uzaylar tanımlanmaktadır. Ayrıca asimetrik konik normlu uzaylarda bazı topolojik kavramlar tanıtılmaktadır ve çalışılmaktadır.

Anahtar kelimeler: Asimetrik norm, konik norm, yakınsaklık, tamlik.

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1. Introduction and preliminaries

In [1], the authors introduced the cone metric spaces by means of a partial ordering on a Banach space with a cone. By defining convergence and completeness in these spaces, they prove some fixed point theorems to generalize the corresponding ones in metric spaces. Later, the authors of [2] gave some generalized topological concepts and definitions in cone metric spaces and proved that every cone metric space is a topological space.

As a generalization of a norm, a cone norm is defined in [3,4] by replacing the set of real numbers with an ordered real Banach space. A real vector space with a cone norm is called a cone normed space. Cone normed spaces play an important role in fixed point theory, computer science and some other research areas of functional analysis.

Before giving the formal definition of a cone norm, we give some basic notions and results related to the topic.

**Definition 1.** [5] Let $P$ be a subset of a real Banach space $E$. Then $P$ is called a cone if the following conditions hold:

- $P$ is closed, $P \neq \{\theta_E\}$ and $P \neq \emptyset$,
- $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geq 0$ and $x, y \in P \Rightarrow \alpha x + \beta y \in P$,
- $x \in P$ and $-x \in P \Rightarrow x = \theta_E$, where $\theta_E$ denotes the zero vector of the real vector space $E$.

**Example 1.** [5]

1. Let $E = \mathbb{R}^n$. Then $P = \{(x_1, \ldots, x_n) \in E: x_i \geq 0 \text{ for all } i = 1, \ldots, n\}$ is a cone on $E$.
2. Let $E = C[a, b]$. Then $P = \{f \in E: f(x) \geq 0 \text{ for all } x \in [a, b]\}$ is a cone on $E$.
3. Let $E = \ell_p$ $(1 \leq p < \infty)$. Then $P = \{(x_n) \in E: x_n \geq 0 \text{ for all } n \in \mathbb{N}\}$ is a cone on $E$.

Let $P$ be a cone on a real Banach space $E$. For any $x, y \in E$, we mean

1. $x \leq y \iff y - x \in P$,
2. $x < y \iff y - x \in P, x \neq y$,
3. $x \ll y \iff y - x \in \text{Int}P$, where $\text{Int}P$ denotes the interior of $P$.

It is easy to see that $\leq$ defines a partial ordering on $E$ with respect to $P$. The cone $P$ is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y \Rightarrow \|x\| \leq K \|y\|$, where $K$ is called the normal constant of $P$. $P$ is called minihedral cone if for any $x, y \in E$, $\sup \{x, y\}$ exists, or equivalently $\inf \{x, y\}$ exists.

**Lemma 1.** [2] Let $P$ be a cone on a real Banach space $E$. For every $c \in E$ with $\theta_E \ll c$, there exists $\delta > 0$ such that $u \ll c$ whenever $u \in E$ with $\|u\| < \delta$.

**Lemma 2.** [2] Let $P$ be a cone on a real Banach space $E$. For every $c_1, c_2 \in E$ with $\theta_E < c_1, c_2$, there exists $c \in E$ with $\theta_E \ll c$ such that $c \ll c_1, c_2$.

**Lemma 3.** [6] Let $P$ be a cone on a real Banach space $E$. If $\theta_E \leq u \ll c$ for every $c \in E$. 

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$E$ with $\theta_E \ll c$, then $u = \theta_E$.

In our proofs, we use the following two facts:

$$\lambda \in \mathbb{R}, \lambda \geq 0, x \leq y \Rightarrow \lambda x \leq \lambda y.$$  \hspace{1cm} (1)

$$\lambda \in \mathbb{R}, \lambda > 0 \Rightarrow \lambda \text{Int} P \subset \text{Int} P.$$  \hspace{1cm} (2)

Throughout the study, $(E, \| \|)$ is a real Banach space and $P$ is a cone on $E$ with $\text{Int} P \neq \emptyset$.

**Definition 2.** [3,4] A cone norm on a real vector space $X$ is a mapping $\| \|_c: X \rightarrow E$ such that

$$\theta_E \leq \| x \|_c$$

$$\| x \|_c = \theta_E \iff x = \theta_X$$

$$\| \alpha x \|_c = |\alpha| \| x \|_c$$

$$\| x + y \|_c \leq \| x \|_c + \| y \|_c$$

hold for all $x, y \in X$ and $\alpha \in \mathbb{R}$, where $\theta_X$ denotes the zero vector of $X$. The ordered pair $(X, \| \|_c)$ is called a cone normed space.

The study of asymmetric metrics goes back to Wilson [7] and then it became a subject of intensive research in the context of topology and theoretical computer science. Following his terminology, asymmetric metric is often called quasi metric. In the realms of pure and applied mathematics and materials science, there are many applications of asymmetric metric spaces.

An asymmetric norm is a positive definite sublinear functional on a real vector space. The definition is as follows:

**Definition 3.** [8] An asymmetric norm on real vector space $X$ is a mapping $p: X \rightarrow \mathbb{R}$ such that

$$p(x) \geq 0$$

$$p(x) = p(-x) = 0 \iff x = \theta_X$$

$$p(\alpha x) = \alpha p(x)$$

$$p(x + y) \leq p(x) + p(y)$$
hold for all \(x, y \in X\) and \(\alpha \geq 0\). The ordered pair \((X, p)\) is called an asymmetric normed space.

On the contrary to a norm, since the scalar multiplication is not continuous, an asymmetric norm does not induce a vector topology. An asymmetric norm defines an asymmetric metric which does not satisfy the symmetry condition of a metric. Hence one can obtain a topology induced by the asymmetric norm which is not necessarily Hausdorff. This innocent modification changes the whole theory, mainly related to completeness, compactness and totally boundedness. For instance, sequentially compactness and compactness are not the same notions contrary to the case of a normed space. Many authors have investigated the topological properties of asymmetric metric and related structures. We refer to [9-15] and references therein.

In this study, we give the definition of an asymmetric cone normed space as a generalization of the asymmetric normed space. Also, we introduce some topological concepts with basic results on asymmetric cone normed spaces.

2. Main results

In this section, we define asymmetric cone normed spaces and give some new results related to these spaces.

An asymmetric cone norm on a real vector space \(X\) is a mapping \(p_c:X \rightarrow E\) satisfying the following conditions:

\[
\begin{align*}
\theta_E & \leq p_c(x) \\
p_c(x) &= p_c(-x) = \theta_E \iff x = \theta_X \\
p_c(\alpha x) &= \alpha p_c(x) \\
p_c(x + y) &\leq p_c(x) + p_c(y)
\end{align*}
\]

for all \(x, y \in X\) and \(\alpha \geq 0\). The ordered pair \((X, p_c)\) is called an asymmetric cone normed space.

The mapping \(\bar{p}_c:X \rightarrow E\) defined by \(\bar{p}_c(x) = p_c(-x)\) for all \(x \in X\) is an asymmetric cone norm on \(X\) and called as the conjugate of asymmetric cone norm \(p_c\).

**Lemma 4.** If \(P\) is a minihedral cone on \(E\), then the mapping \(p_c^\#:X \rightarrow E\) defined by \(p_c^\#(x) = \sup\{p_c(x), \bar{p}_c(x)\}\) is a cone norm on \(X\).

**Proof.** Clearly, \(\theta_E \leq p_c(x), \bar{p}_c(x) \leq p_c^\#(x)\) holds for every \(x \in X\).

It can be easily seen that \(p_c^\#(x) = \theta_E \iff p_c(x) = p_c(-x) = \theta_E \iff x = \theta_X\).
Firstly, let $\alpha \geq 0$. By (1), we obtain $\alpha p_c(x), \alpha \tilde{p}_c(x) \leq \alpha p_c^\delta(x)$ and so $p_c(\alpha x), \tilde{p}_c(\alpha x) \leq \alpha p_c^\delta(x)$. This means that

$$p_c^\delta(\alpha x) \leq \alpha p_c^\delta(x). \quad (3)$$

Now, let $\alpha < 0$. It is clear that $p_c^\delta(-\alpha x) = \sup\{p_c(-\alpha x), \tilde{p}_c(-\alpha x)\} = p_c^\delta(\alpha x)$. From (3), we obtain $p_c^\delta(-\alpha x) \leq -\alpha p_c^\delta(x) = |\alpha|p_c^\delta(x)$ or equivalently, $p_c^\delta(\alpha x) \leq |\alpha|p_c^\delta(x)$.

Hence, for $\alpha \neq 0$, we have $|\alpha|p_c^\delta(x) = |\alpha|p_c^\delta(\frac{1}{\alpha}ax) \leq |\alpha||\frac{1}{\alpha}|p_c^\delta(ax) = p_c^\delta(ax)$.

Consequently, it follows that $p_c^\delta(ax) = |\alpha|p_c^\delta(x)$ for all $\alpha \in \mathbb{R}$.

We have for all $x, y \in X$, $p_c(x + y) \leq p_c(x) + p_c(y)$ and $\tilde{p}_c(x + y) \leq \tilde{p}_c(x) + \tilde{p}_c(y)$ which imply together that $p_c^\delta(x + y) \leq p_c^\delta(x) + p_c^\delta(y)$.

For $x \in X$ and $c \in E$ with $\theta_E \ll c$, we define the open and closed balls by:

$$B_{p_c}(x, c) = \{y \in X: p_c(y - x) \ll c\}$$

and

$$B_{p_c}[x, c] = \{y \in X: p_c(y - x) \leq c\},$$

respectively.

For an asymmetric cone norm $p_c$, one can define an asymmetric cone metric $q_{p_c}$ by the formula $q_{p_c}(x, y) = p_c(y - x)$ $(x, y \in X)$. Hence, a topology $\tau_{p_c}$ on $X$ generated by the collection

$$\{B_{p_c}(x, c): x \in X, c \in E \text{ with } \theta_E \ll c\}$$

can be defined. That is, a set $U$ in an asymmetric cone normed space $X$ is open with respect to the topology $\tau_{p_c}$ if for all $x \in U$ there exists $c_x \in E$ with $\theta_E \ll c_x$ such that $B_{p_c}(x, c_x) \subset U$.

In the same way, another topology $\tau_{\tilde{p}_c}$ on $X$ generated by the collection

$$\{B_{\tilde{p}_c}(x, c): x \in X, c \in E \text{ with } \theta_E \ll c\}$$

can be defined. $U$ is open with respect to $\tau_{\tilde{p}_c}$ if for all $x \in U$ there exists $c_x \in E$ with $\theta_E \ll c_x$ such that $B_{\tilde{p}_c}(x, c_x) \subset U$.

**Remark 1.** The topology $\tau_{p_c}$ is not necessarily $T_1$.

**Example 2.** Let $X = \mathbb{R}$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E: x, y \geq 0\}$ and $p_c: X \to E$ defined by
\[ p_c(x) = \begin{cases} (x, x), & \text{if } x > 0 \\ (0,0), & \text{if } x \leq 0. \end{cases} \]

Then, the topology \( \tau_{p_c} \) is not \( T_1 \). In fact, given any \((c_1, c_2) \in E\) with \( c_1, c_2 > 0 \), we have \( 1 \in B_{p_c}(2, (c_1, c_2)) = (-\infty, c) \), where \( c = \min\{c_1 + 2, c_2 + 2\} \). That is, there exists no neighbourhood of 2 with respect to \( \tau_{p_c} \) which does not contain 1. Hence, \( \tau_{p_c} \) cannot be \( T_1 \).

**Theorem 1.** Let \((X, p_c)\) be an asymmetric cone normed space. The topology \( \tau_{p_c} \) is \( T_1 \) if and only if \( \theta_E << p_c(x) \) for every \( x \in X, x \neq \theta_X \).

**Proof.** \( \Rightarrow \) Let \( \tau_{p_c} \) be \( T_1 \). Then, given any \( x \in X \) with \( x \neq \theta_X \), there exist \( c_x, c \in E \) with \( \theta_E << c_x, c \) such that \( \theta_X \notin B_{p_c}(x, c_x) \) and \( x \notin B_{p_c}(\theta_X, c) \); that is, \( \theta_E << c_x \leq p_c(-x) \) and \( \theta_E << c \leq p_c(x) \).

\( \Leftarrow \) By hypothesis, we have \( \theta_E << p_c(y - x) \) and \( \theta_E << p_c(x - y) \) for every \( x, y \in X \) with \( x \neq y \). Then, we obtain that \( y \notin B_{p_c}(x, c_x) \) and \( x \notin B_{p_c}(y, c_y) \), where \( c_x = p_c(y - x) \) and \( c_y = p_c(x - y) \), respectively. Hence, we conclude that \( \tau_{p_c} \) is \( T_1 \).

**Lemma 5.** \( B_{p_c}(x, c) \) is open with respect to \( \tau_{p_c} \).

**Proof.** Let \( y \in B_{p_c}(x, c) \). Then, we have \( p_c(y - x) << c \). Put \( c_y = c - p_c(y - x) \). If \( z \in B_{p_c}(y, c_y) \), we obtain

\[ p_c(z - x) \leq p_c(z - y) + p_c(y - x) << c_y + p_c(y - x) = c \]

which means \( z \in B_{p_c}(x, c) \). Hence, we conclude that \( B_{p_c}(x, c) \) is open with respect to \( \tau_{p_c} \) since given any \( y \in B_{p_c}(x, c) \) the inclusion \( B_{p_c}(y, c_y) \subset B_{p_c}(x, c) \) holds for some \( c_y \in E \) with \( \theta_E << c_y \).

**Lemma 6.** \( B_{p_c}[x, c] \) is closed with respect to \( \tau_{p_c} \).

**Proof.** Let \( y \in X \setminus B_{p_c}[x, c] \). Then, we have \( c << p_c(y - x) \). Put \( c_y = p_c(y - x) - c \). If \( z \in B_{p_c}(y, c_y) \), we obtain

\[ p_c(y - x) \leq p_c(y - z) + p_c(z - x) << c_y + p_c(z - x) \]

which implies \( c = p_c(y - x) - c_y << p_c(z - x) \) and so \( z \in X \setminus B_{p_c}[x, c] \). Hence, \( X \setminus B_{p_c}[x, c] \) is open with respect to \( \tau_{\bar{p}_c} \); that is, \( B_{p_c}[x, c] \) is closed with respect to \( \tau_{p_c} \).

**Lemma 7.** In an asymmetric cone normed space \((X, p_c)\), \( B_{p_c}(x, c) = x + \| c \| B_{p_c}(\theta_E, e) \) and \( B_{p_c}[x, c] = x + \| c \| B_{p_c}[\theta_E, e] \) hold for every \( x \in X \) and \( c \in E \) with \( \theta_E << c \), where
\[ e = \frac{c}{\|c\|}. \]

**Proof.** Let \( y \in B_{p_{c}}(x, c) \). Then, we have \( p_{c}(y - x) << c \). From (2), we obtain
\[ p_{c}\left(\frac{1}{\|c\|}(y - x)\right) << \frac{c}{\|c\|} \] which means that \( \frac{1}{\|c\|}(y - x) \in B_{p_{c}}(\theta_{E}, e) \) and so \( y \in x + \|c\| B_{p_{c}}(\theta_{E}, e) \), where \( e = \frac{c}{\|c\|} \). Hence, we conclude that \( B_{p_{c}}(x, c) \subset x + \|c\| B_{p_{c}}(\theta_{E}, e) \).

For the reverse inclusion, let \( y \in x + \|c\| B_{p_{c}}(\theta_{E}, e) \) (\( e = \frac{c}{\|c\|} \)). Then, we write \( y = x + \|c\| z \) for \( z \in E \) with \( p(z) << \frac{c}{\|c\|} \). It follows that \( p(y - x) = \|c\| p(z) << c \) and so \( y \in B_{p_{c}}(x, c) \). Consequently, the inclusion \( x + \|c\| B_{p_{c}}(\theta_{E}, e) \subset B_{p_{c}}(x, c) \) holds.

**Theorem 2.** The mapping \( p_{c} \) is upper semi continuous and lower semi continuous with respect to the topologies \( \tau_{p_{c}} \) and \( \bar{\tau}_{p_{c}} \) respectively.

**Proof.** Let \( A = \{x \in X : p_{c}(x) << u\} \) for any \( u \in E \). Choose \( x \in A \) and put \( c = u - p_{c}(x) \) which satisfies \( \theta_{E} << c \). For \( z \in B_{p_{c}}(x, c) \), we have
\[ p_{c}(z) \leq p_{c}(z - x) + p_{c}(x) << c + p_{c}(x) = u \]
which means \( z \in A \). Hence, the inclusion \( B_{p_{c}}(x, c) \subset A \) holds. We conclude that \( A \) is \( \tau_{p_{c}} \)-open and so \( p_{c} \) is upper semi continuous with respect to \( \tau_{p_{c}} \).

Let \( B = \{x \in X : u << p_{c}(x)\} \) for any \( u \in E \). Choose \( x \in B \) and put \( c' = p_{c}(x) - u \) which satisfies \( \theta_{E} << c' \). For \( z \in B_{\bar{p}_{c}}(x, c') \), we have
\[ p_{c}(x) \leq p_{c}(x - z) + p_{c}(z) = \bar{p}_{c}(z - x) + p_{c}(z) << c' + p_{c}(z) \]
and therefore
\[ u = p_{c}(x) - c' << p_{c}(z) \]
which means \( z \in B \). Hence, the inclusion \( B_{\bar{p}_{c}}(x, c') \subset B \) holds. We conclude that \( B \) is \( \tau_{\bar{p}_{c}} \)-open and so \( p_{c} \) is lower semi continuous with respect to \( \tau_{\bar{p}_{c}} \).

**Definition 4.** A sequence \( (x_{n}) \) in an asymmetric cone normed space \((X, p_{c})\) is said to be left (right) \( p_{c} \)-convergent to \( x \in X \) if for every \( c \in E \) with \( \theta_{E} << c \), there exists \( n_{c} \in \mathbb{N} \) such that \( p_{c}(x_{n} - x) << c \) (\( p_{c}(x - x_{n}) << c \)) for all \( n \geq n_{c} \). We denote it by \( x_{n} \xrightarrow{\ell}^{r} x \) \((x_{n} \xrightarrow{\ell}^{r} x)\).

**Remark 2.** A sequence \( (x_{n}) \) in \((X, p_{c})\) is convergent to \( x \in X \) with respect to the cone norm \( p_{c}^{s} \) if and only if it is left \( p_{c} \)-convergent and right \( p_{c} \)-convergent to \( x \).

**Lemma 8.** Let \( (x_{n}) \) be a sequence in an asymmetric cone normed space \((X, p_{c})\).
1. If \((x_n)\) is left \(p_c\)-convergent to \(x \in X\) and right \(p_c\)-convergent to \(y \in X\), then \(p_c(y - x) = \theta_E\).

2. If \((x_n)\) is left \(p_c\)-convergent to \(x \in X\) and \(p_c(x - y) = \theta_E\), then \((x_n)\) is also left \(p_c\)-convergent to \(y \in X\).

**Proof.** By hypothesis, given any \(c \in E\) with \(\theta_E \ll c\) there exists \(n_c \in \mathbb{N}\) such that \(p_c(x_n - x) \ll \frac{c}{2}\) and \(p_c(y - x_n) \ll \frac{c}{2}\) for all \(n \geq n_c\). Hence, we obtain

\[
\theta_E \leq p_c(y - x) \leq p_c(y - x_n) + p_c(x_n - x) \ll c.
\]

From Lemma 3, we conclude that \(p_c(y - x) = \theta_E\).

By hypothesis, given any \(c \in E\) with \(\theta_E \ll c\), there exists \(n_c \in \mathbb{N}\) such that

\[
p_c(x_n - y) \leq p_c(x_n - x) + p_c(x - y) \ll c
\]

for all \(n \geq n_c\) which means that \((x_n)\) is left \(p_c\)-convergent to \(y\).

**Remark 3.** As a result, it is clear that if a sequence \((x_n)\) in an asymmetric cone normed space \((X, p_c)\) is left \(p_c\)-convergent to \(x \in X\), then \(x\) is not unique unlike in a cone normed space.

**Lemma 9.** Let \((X, p_c)\) be an asymmetric cone normed space with a normal cone \(P\) on \(E\) and \((x_n)\) be a sequence in \(X\). \((x_n)\) is left \(p_c\)-convergent to \(x \in X\) if and only if \(p_c(x_n - x) \to \theta_E\) as \(n \to \infty\).

**Proof.** \(\Rightarrow\) Let \(x_n \overset{l}{\to} x\). Given any \(\varepsilon > 0\), choose \(c \in E\) with \(\theta_E \ll c\) such that \(K \parallel c \parallel < \varepsilon\), where \(K > 0\) is a normal constant. Then, there is \(n_0 \in \mathbb{N}\) satisfying \(\parallel p_c(x_n - x) \parallel \leq K \parallel c \parallel \ll \varepsilon\) for all \(n \geq n_0\) which means \(p_c(x_n - x) \to \theta_E\).

\(<\) Suppose that \(p_c(x_n - x) \to \theta_E\). From Lemma 1, for \(c \in E\) with \(\theta_E \ll c\), there exists \(\delta > 0\) such that \(a \ll c\) whenever \(\parallel a \parallel \ll \delta\). For this \(\delta > 0\), there is \(n_0 \in \mathbb{N}\) satisfying \(\parallel p_c(x_n - x) \parallel < \delta\) for all \(n \geq n_0\). It follows that \(p_c(x_n - x) \ll c\) for all \(n \geq n_0\) and so \((x_n)\) is left \(p_c\)-convergent to \(x\).

**Example 3.** Let \(X = \mathbb{R}\), \(E = \mathbb{R}^2\), \(P = \{(x, y) \in E : x, y \geq 0\}\) and \(p_c : X \to E\) defined by \(p_c(x) = \begin{cases} (x, x), & \text{if } x > 0 \\ (0, 0), & \text{if } x \leq 0. \end{cases}\)

The sequence \((-1)^n\) is left \(p_c\)-convergent to 1. Indeed, given any \((c_1, c_2) \in E\) with \((0, 0) \ll (c_1, c_2)\), we have \(p_c((-1)^n - 1) = (0, 0)\) for all \(n \in \mathbb{N}\). (Clearly, this sequence is not only left \(p_c\)-convergent to 1 but also left \(p_c\)-convergent to all \(x > 1\).) Also, this sequence is right \(p_c\)-convergent to \(-1\) since \(p_c(-1 - (-1)^n) = (0, 0)\) for all \(n \in \mathbb{N}\). By using Lemma 9, we conclude that it is not convergent with respect to the cone norm \(p_c\) since we have...
\[ p_c^\delta((-1)^n - x) = \max(\max(0, -\frac{1}{2}((-1)^n - x)), \max(0, \frac{1}{2}((-1)^n - x))) \Rightarrow (0,0) \]

for any \( x \in X \).

**Definition 5.** A sequence \((x_n)\) in an asymmetric cone normed space \((X, p_c)\) is said to be
1. \(p_c^\delta\)-Cauchy if for every \(c \in E\) with \(\theta_E \ll c\) there exists \(n_c \in \mathbb{N}\) such that \(p_c(x_n - x_m) \ll c\) for all \(n, m \geq n_c\).
2. left (right) \(K\)-Cauchy if for every \(c \in E\) with \(\theta_E \ll c\) there exists \(n_c \in \mathbb{N}\) such that \(p_c(x_n - x_m) \ll c\) (\(p_c(x_m - x_n) \ll c\)) for all \(n \geq m \geq n_c\).
3. weakly left (right) \(K\)-Cauchy if for every \(c \in E\) with \(\theta_E \ll c\) there exists \(n_c \in \mathbb{N}\) such that \(p_c(x_n - x_{n_c}) \ll c\) (\(p_c(x_{n_c} - x_n) \ll c\)) for all \(n \geq n_c\).
4. left (right) \(p_c\)-Cauchy if for every \(c \in E\) with \(\theta_E \ll c\) there exist \(n_c \in \mathbb{N}\) and \(x \in X\) such that \(p_c(x_n - x) \ll c\) (\(p_c(x - x_n) \ll c\)) for all \(n \geq n_c\).

**Remark 4.** A sequence \((x_n)\) in \((X, p_c)\) is \(p_c^\delta\)-Cauchy if and only if it is left \(K\)-Cauchy and right \(K\)-Cauchy.

**Remark 5.** The following relations hold:
\(p_c^\delta\)-Cauchy \(\Rightarrow\) left (right) \(K\)-Cauchy \(\Rightarrow\) weakly left (right) \(K\)-Cauchy \(\Rightarrow\) left (right) \(p_c\)-Cauchy

**Example 4.** Let \(X = \mathbb{R}, E = \mathbb{R}^2, P = \{(x, y) \in E : x, y \geq 0\}\) and \(p_c : X \rightarrow E\) defined by
\[ p_c(x) = \begin{cases} (1,1), & \text{if} x > 0 \\ (0,0), & \text{if} x \leq 0. \end{cases} \]

The sequence \((x_n) = (1,0,1,0,\ldots)\) is weakly left \(K\)-Cauchy. Indeed, given any \((c_1, c_2) \in E\) with \((0,0) \ll (c_1, c_2)\), we have \(p_c(x_n - x_1) = (0,0) \ll (c_1, c_2)\) for all \(n \geq 1\). But, it is not left \(K\)-Cauchy due to the fact that for \(n > m\), we have \(p_c(x_n - x_m) = (1,1) \ll (\frac{1}{2}, \frac{1}{2})\), where \(n\) is odd and \(m\) is even.

**Definition 6.** Let \((X, p_c)\) be an asymmetric cone normed space.
1. If every left \(p_c\)-Cauchy sequence in \(X\) is left \(p_c\)-convergent, then \(X\) is called left \(p_c\)-sequentially complete asymmetric cone normed space.
2. If every weakly left \(K\)-Cauchy sequence in \(X\) is left \(p_c\)-convergent, then \(X\) is called weakly left \(K\)-sequentially complete asymmetric cone normed space.
3. If every left \(K\)-Cauchy sequence in \(X\) is left \(p_c\)-convergent, then \(X\) is called left \(K\)-sequentially complete asymmetric cone normed space.
4. If every \(p_c^\delta\)-Cauchy sequence in \(X\) is left \(p_c\)-convergent, then \(X\) is called \(p_c\)-sequentially complete asymmetric cone normed space.

**Theorem 3.** Let \((x_n)\) be a left \(K\)-Cauchy sequence in an asymmetric cone normed space \((X, p_c)\).
1. If \((x_n)\) has a left \(p_c\)-convergent subsequence, then \((x_n)\) is left \(p_c\)-convergent to the
same point.
2. If \((x_n)\) has a right \(p_c\)-convergent subsequence, then \((x_n)\) is right \(p_c\)-convergent to the same point.

**Proof.** 1. Suppose that \((x_n)\) is a left \(K\)-Cauchy sequence in \((X, p_c)\) and the subsequence \((x_{n_k})\) is left \(p_c\)-convergent to \(x \in X\). Then, given any \(c \in E\) with \(\theta_E \ll c\) there exists \(n_c \in \mathbb{N}\) such that \(p_c(x_n - x_m) \ll \frac{c}{2}\) for all \(n \geq m \geq n_c\). Choose \(k_0 \in \mathbb{N}\) such that \(n_{k_0} \geq n_c\) and \(p_c(x_{n_k} - x) \ll \frac{c}{2}\) for all \(k \geq k_0\). It follows that

\[
p_c(x_n - x) \leq p_c(x_n - x_{n_{k_0}}) + p_c(x_{n_{k_0}} - x) \ll c
\]

for \(n \geq n_{k_0}\).

2. Now, suppose that the subsequence \((x_{n_k})\) is right \(p_c\)-convergent to \(x \in X\). Choose \(k_0 \in \mathbb{N}\) such that \(n_k \geq n \geq n_c\) and \(p_c(x - x_{n_k}) \ll \frac{c}{2}\) for all \(k \geq k_0\). Then, we have

\[
p_c(x - x_n) \leq p_c(x - x_{n_k}) + p_c(x_{n_k} - x) \ll c.
\]

**Corollary 1.** If \((x_n)\) is a left \(K\)-Cauchy sequence in an asymmetric cone normed space \((X, p_c)\) and the subsequence \((x_{n_k})\) is convergent to \(x \in X\) with respect to the cone norm \(p_c^\delta\), then \((x_n)\) is convergent to \(x \in X\) with respect to the cone norm \(p_c^\delta\).

**Proof.** Let \((x_n)\) be a left \(K\)-Cauchy sequence in \((X, p_c)\). Suppose that there is a subsequence \((x_{n_k})\) of \((x_n)\) such that \((x_{n_k})\) is convergent to \(x \in X\) with respect to the cone norm \(p_c^\delta\). By Remark 2, we have \(x_{n_k} \overset{l}{\longrightarrow} x\) and \(x_{n_k} \overset{r}{\longrightarrow} x\). The last theorem implies that \(x_n \overset{l}{\longrightarrow} x\) and \(x_n \overset{r}{\longrightarrow} x\). Again by Remark 2, it follows that \((x_n)\) is convergent to \(x\) with respect to \(p_c^\delta\).

**Theorem 4.** Let \((x_n)\) be a sequence in an asymmetric cone normed space \((X, p_c)\). If

\[
\sum_{n=1}^{\infty} \| p_c(x_{n+1} - x_n) \| < \infty
\]

holds, then \((x_n)\) is a left \(K\)-Cauchy sequence in \(X\).

**Proof.** Let \(c \in E\) with \(\theta_E \ll c\). From Lemma 1, we can find a \(\delta > 0\) such that \(a \ll c\) holds for every \(a \in E\) with \(\| a \| < \delta\). By hypothesis, for this \(\delta > 0\), there exists \(n_0 \in \mathbb{N}\) such that

\[
\sum_{i=0}^{\infty} \| p_c(x_{n_0+i+1} - x_{n_0+i}) \| < \delta.
\]
Hence, we have

\[ \| p_c(x_{n+k} - x_n) \| \leq \sum_{i=0}^{k-1} \| p_c(x_{n+i+1} - x_{n+i}) \| < \delta \]

which implies that \( p_c(x_{n+k} - x_n) << c \) for all \( n \geq n_0 \) and \( k \in \mathbb{N} \). This means that \((x_n)\) is a left \( K\)-Cauchy sequence in \( X \).

**Theorem 5.** Let \( P \) be a normal cone with normal constant \( K \). The asymmetric cone normed space \((X, p_c)\) is left \( K\)-sequentially complete if and only if the sequence \((X_n) = (x_1 + x_2 + \ldots + x_n)\) is left \( p_c\)-convergent in \( X \) whenever \((x_n)\) is a sequence in \( X \) such that \( \sum_{n=1}^{\infty} \| p_c(x_n) \| < \infty \).

**Proof.** (⇒) Suppose that \( X \) is left \( K\)-sequentially complete. Given any \( c \in E \) with \( \theta_E << c \), choose \( \delta > 0 \) as in the proof of the previous theorem. If \((x_n)\) is a sequence in \( X \) such that \( \sum_{n=1}^{\infty} \| p_c(x_n) \| < \infty \) holds, then there exists \( n_0 \in \mathbb{N} \) satisfying \( \sum_{i=0}^{\infty} \| p_c(x_{n_0+i}) \| < \delta \). Hence, we have

\[ \| p_c(x_{n+k} - x_n) \| \leq \sum_{i=1}^{k} \| p_c(x_{n+i}) \| < \delta \]

which implies that \( p_c(x_{n+k} - x_n) << c \) for all \( n \geq n_0 \) and \( k \in \mathbb{N} \). This means that \((X_n)\) is a left \( K\)-Cauchy sequence in \( X \). By left \( K\)-sequentially completeness of \( X \), the sequence \((X_n)\) is left \( p_c\)-convergent in \( X \).

(⇐) Take a left \( K\)-Cauchy sequence \((x_n)\) in \( X \). Then, for \( e \in \text{Int} P \) with \( \| e \| = 1 \), there exists \( n_1 \in \mathbb{N} \) such that \( p_c(x_n - x_m) << \frac{e}{2} \) for all \( n \geq m \geq n_1 \). Choose \( n_2 \in \mathbb{N} \) such that \( n_2 > n_1 \) and \( p_c(x_n - x_m) << \frac{e}{2^2} \) for all \( n \geq m \geq n_2 \). Continuing in this manner, we obtain an increasing sequence of natural numbers \( n_1 < n_2 < \ldots \) satisfying \( p_c(x_{n_{i+1}} - x_{n_i}) << \frac{e}{2^i} \) for all \( i \in \mathbb{N} \). Hence, we obtain \( \sum_{i=1}^{\infty} \| p_c(x_{n_{i+1}} - x_{n_i}) \| \leq K < \infty \). By hypothesis, the sequence \(((x_{n_2} - x_{n_1}) + (x_{n_3} - x_{n_2}) + \ldots + (x_{n_k+1} - x_{n_k})) = (x_{n_k+1} - x_{n_2})\) is left \( p_c\)-convergent to some \( y \in X \) which implies that the subsequence \((x_{n_k})\) is left \( p_c\)-convergent to \( y + x_{n_1} \). From Theorem 3, \((x_n)\) is also left \( p_c\)-convergent to \( y + x_{n_1} \) and so \( X \) is left \( K\)-sequentially complete.

**Theorem 6.** An asymmetric cone normed space \((X, p_c)\) is weakly left \( K\)-sequentially complete if and only if it is left \( K\)-sequentially complete.

**Proof.** (⇒) Since a left \( K\)-Cauchy sequence is also a weakly left \( K\)-Cauchy sequence, then it is obvious that weakly left \( K\)-sequentially completeness of \( X \) implies its left \( K\)-sequentially completeness.
Now, suppose that $X$ is left $K$-sequentially complete asymmetric cone normed space. Let $(x_n)$ be a weakly left $K$-Cauchy sequence in $X$. For $e \in \text{Int}P$ with $\|e\|=1$, choose the smallest natural number $n_1$ satisfying $p_c(x_n - x_{n_1}) \ll e$ for all $n \geq n_1$. If $p_c(x_n - x_{n_1}) = \theta_E$ for all $n \geq n_1$, then we have $x_n \rightarrow x_{n_1}$ which completes the proof.

If $\theta_E \ll p_c(x_{m_1} - x_{n_1})$ for some $m_1 > n_1$, then from Lemma 2 there exists $k_2 \in \mathbb{N}$ such that

$$\frac{e}{k_2} \ll p_c(x_{m_1} - x_{n_1}) \ll e.$$ 

Let $n_2$ be the smallest natural number satisfying $p_c(x_n - x_{n_2}) \ll \frac{e}{k_2}$ for all $n \geq n_2$.

Similarly, suppose that $\theta_E \ll p_c(x_{m_2} - x_{n_2})$ for some $m_2 > n_2$, then from Lemma 2 there exists $k_3 \in \mathbb{N}$ such that

$$\frac{e}{k_3} \ll p_c(x_{m_2} - x_{n_2}) \ll \frac{e}{k_2}.$$ 

By continuing the same process, we obtain increasing sequences of natural numbers $1 = k_1 < k_2 < \ldots$ and $n_1 < n_2 < \ldots$ such that $p_c(x_n - x_{n_i}) \ll \frac{e}{k_i}$ for all $n \geq n_i$ and $i \in \mathbb{N}$. The subsequence $(x_{n_i})$ constructed in this way is a left $K$-Cauchy sequence. In fact, given any $c \in E$ with $\theta_E \ll c$, we can find $i_0 \in \mathbb{N}$ such that $\frac{e}{k_{i_0}} \ll c$ and so $p_c(x_{n_i} - x_{n_j}) < c$ for all $i \geq j \geq i_0$. Since $X$ is left $K$-sequentially complete, $(x_{n_i})$ is left $p_c$-convergent to some $x \in X$. Also,

$$p_c(x_n - x) \leq p_c(x_n - x_{n_i}) + p_c(x_{n_i} - x) \ll c$$

holds for sufficiently large $i \in \mathbb{N}$. Hence, we conclude that given any weakly left $K$-Cauchy sequence in $X$ is left $p_c$-convergent which means $X$ is weakly left $K$-sequentially complete.

References


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