Double reduction of second order Benjamin-Ono equation via conservation laws and the exact solutions

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Abstract

In this study, the Benjamin-Ono equation which was first introduced to describe internal waves in stratified fluids are considered. Using the association between Lie point symmetries and local conserved vectors, a reduction in both the number of variables and the order of the equation is achieved. The auxiliary equation method successfully applied to the reduced equation and different types of solutions are obtained. Moreover, some graphical representations for special values of the parameters in solutions are presented.

Keywords: Double reduction method, conservation vectors, Benjamin-Ono equation.

İkinci mertebeden Benjamin-Ono denkleminin korunum kanunları yardımıyla çift indirgemesi ve tam çözümleri

Öz

Bu çalışmada, ilk kez tabakalı sıvılardaki iç dalgaları tanımlamak için sunulan Benjamin-Ono denkləmini ele alınmıştır. Lie nokta simetrileri ve yerel korunum vektörleri arasındaki ilişkiyi kullanarak hem değişken sayısında hem de denklem mertebesinde bir indirgeme elde edilmiştir. İndirgenen denkleme yardımcı denklem metodu başarıyla bir şekilde uygulannmış ve farklı tipte çözümler elde edilmiştir. Ayrıca çözümlerdeki parametrelerin özel değerleri için bazı grafik temsilleri verilmiştir.

Anahtar kelimeler: Çift indirgeme yöntemi, korunum vektörleri, Benjamin-Ono denklemi.

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1. Introduction

Nonlinear evolution equations occur not only in many areas of mathematics but also in other disciplines such as biology, engineering sciences, space sciences, physics, quantum mechanics, chemistry and materials science. There are many nonlinear evolution equations naturally arising from various branches of science such as the nonlinear Sine-Gordon equation in quantum mechanics [1], the Cahn-Hilliard equation in the study of phase transitions of binary alloys [2], the Navier-Stokes equations in the study of the flow of viscous incompressible fluids [3].

Thanks to advances in computer science, using software programs (e.g. Maple, Mathematica), a number of useful methods and theories have been developed and implemented to find solutions to nonlinear evolution equations [4-23].

In this paper, we consider the second order Benjamin-Ono equation [24, 25]

\[ u_{tt} + \beta(u^2)_{xx} + \gamma u_{xxxx} = 0, \]

which is presented to model the percolation of water on the porous surface of a horizontal layer of material, as well as the analysis of long waves in shallow water. In Eq. (1), dependent variable is the elevation of the free surface of the fluid; the vertical deflection or the quadratic nonlinearity accounts for the curvature of the bending beam, \( \gamma \) is the fluid depth, \( \beta \) is a constant controlling nonlinearity and the characteristic speed of the long waves [26]. Many researches have been conducted on this equation, which has attracted the attention of researchers for many years [27-31].

This paper is structured as follows: In Section 2 and 3, we introduce some properties of the double reduction and auxiliary equation method, respectively. In Section 4, we apply these methods to find the solutions of underlying equation. Our discussions and conclusions are given in Sections 5 and 6, respectively.

2. Overview of double reduction method

Here, the relationships between Lie symmetries and conservation laws of systems of partial differential equations (PDEs) will be presented. Then, how to perform double reduction of the equation under consideration will be introduced.

2.1 Fundamental theorems

Let’s examine the \( sth \) order system of PDEs of \( m \) independent variables \( x = (x^1, x^2, \ldots, x^m) \) and \( n \) dependent variables \( u = (v_1, v_2, \ldots, v_n) \)

\[ P^\alpha(x, v, v_1, \ldots, v_s) = 0, \quad \alpha = 1, \ldots, n, \]

where \( v_{(1)}, v_{(2)}, \ldots, v_{(s)} \) symbolize the first, second, \ldots, \( sth \) order partial derivatives, i.e., \( v_1^\alpha = D_1(v^\alpha), v_2^\alpha = D_1D_1(v^\alpha), \ldots \) respectively, with the total differentiation operator with respect to \( x^i \) given by

\[ D_i = \frac{\partial}{\partial x^i} + v_1^\alpha \frac{\partial}{\partial v^\alpha} + v_2^\alpha \frac{\partial}{\partial v^\alpha} + \ldots, \quad i = 1, \ldots, n, \]
where the Einstein’s summation convention is utilised. The following definitions are acknowledged (see, e.g. [32-34]). The variational operator given by

\[
\frac{\delta}{\delta \nu^\alpha} = \frac{\partial}{\partial \nu^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1 \ldots i_s} \frac{\partial}{\partial \nu^\alpha_{i_1 i_2 \ldots i_s}}, \quad \alpha = 1, \ldots, m.
\] (4)

The Lie-Bäcklund operator is given as

\[
\Gamma = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial \nu^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{S},
\] (5)

where \(\mathcal{S}\) is the space of differential functions. The operator (5) is an abbreviated version of the infinite formal sum

\[
\Gamma = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial \nu^\alpha} + \sum_{s \geq 1} \zeta^\alpha_{i_1 i_2 \ldots i_s} \frac{\partial}{\partial \nu^\alpha_{i_1 i_2 \ldots i_s}},
\] (6)

where the extension coefficients are given by the extension formulae

\[
\zeta^\alpha_{i_1 i_2 \ldots i_s} = D_{i_1 i_2 \ldots i_s} \psi^\alpha + \xi^j \psi^\alpha_{i_1 i_2 \ldots i_s} \quad s > 1,
\] (7)

where \(\psi^\alpha\) is the Lie characteristic function

\[
\psi^\alpha = \eta^\alpha - \xi^j \psi^\alpha_j.
\] (8)

The \(N^i\) Noether operator is presented in terms of \(\Gamma\) operator as

\[
N^i = \xi^i + \psi^\alpha \frac{\partial}{\partial \nu^\alpha} + \sum_{s \geq 1} D_{i_1 i_2 \ldots i_s} \psi^\alpha \frac{\partial}{\partial \nu^\alpha_{i_1 i_2 \ldots i_s}}, \quad i = 1, \ldots, m,
\] (9)

where the variational operators w.r.t. derivatives of \(\nu^\alpha\) are obtained from (4) by replacing \(\nu^\alpha\) by the corresponding derivatives. The \(m\)-tuple vector \(T = (T^1, T^2, \ldots, T^m)\), \(T^j \in S, \quad j = 1, \ldots, m\) is a conserved vector of (2) if \(T^i\) satisfies

\[
D_i T^i \big|_{(2)} = 0.
\] (10)

We now give the relevant results used in this study below.

**Definition** [35]: If the \(T^i\) conserved vectors and \(\Gamma\) operator of the equation (2) satisfy the following expression

\[
\Gamma(T^i) + T^i D_k (\xi^k) - T^k D_k (\xi^i) = 0, \quad i = 1, \ldots, m,
\] (11)

then it is said to be they are associated.

**Theorem** [34, 36]: Assume that \(\Gamma\) is any Lie-Bäcklund operator of Eq. (2) and the components of conserved vector of (2) are given by \(T^i\). Then

\[
T^*i = [T^i, \Gamma] = \Gamma(T^i) + T^i D_j \xi^j - T^j D_j \xi^i, \quad i = 1, \ldots, m,
\] (12)
construct the components of a conserved vector of (2), i.e., \( D_i T^{*i}_{(2)} = 0 \).

**Theorem [37]**: Assume that \( D_i T^i = 0 \) is a conservation law of the PDE system (2). Then under a similarity transformation, there exists functions \( \tilde{T}^i \) such that
\[
J D_i T^i = \tilde{D}_i \tilde{T}^i
\]
where \( \tilde{T}^i \) is given by
\[
\begin{pmatrix}
\tilde{T}^1 \\
\tilde{T}^2 \\
\vdots \\
\tilde{T}^m
\end{pmatrix}
= J (A^{-1})^T \begin{pmatrix}
T^1 \\
T^2 \\
\vdots \\
T^m
\end{pmatrix},
\]
\[
J = \text{det}(A).
\]
in which
\[
A = \begin{pmatrix}
\tilde{D}_1 x^1 & \tilde{D}_1 x^2 & \cdots & \tilde{D}_1 x^m \\
\tilde{D}_2 x^1 & \tilde{D}_2 x^2 & \cdots & \tilde{D}_2 x^m \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{D}_m x^1 & \tilde{D}_m x^2 & \cdots & \tilde{D}_m x^m
\end{pmatrix},
\quad
A^{-1} = \begin{pmatrix}
D_1 \tilde{x}^1 & D_1 \tilde{x}^2 & \cdots & D_1 \tilde{x}^m \\
D_2 \tilde{x}^1 & D_2 \tilde{x}^2 & \cdots & D_2 \tilde{x}^m \\
\vdots & \vdots & \ddots & \vdots \\
D_m \tilde{x}^1 & D_m \tilde{x}^2 & \cdots & D_m \tilde{x}^m
\end{pmatrix}
\]

Theorem([37]): Assume that \( D_i T^i = 0 \) is a conservation law of (2). Then under a similarity transformation of a symmetry \( \Gamma \) (6), there exist functions \( \tilde{T}^i \) such that the symmetry \( \Gamma \) is still a symmetry for the PDE \( \tilde{D}_i \tilde{T}^i \) and
\[
\begin{pmatrix}
\Gamma \tilde{T}^1 \\
\Gamma \tilde{T}^2 \\
\vdots \\
\Gamma \tilde{T}^m
\end{pmatrix}
= J (A^{-1})^T \begin{pmatrix}
[T^1, \Gamma] \\
[T^2, \Gamma] \\
\vdots \\
[T^m, \Gamma]
\end{pmatrix}.
\]

If the conservation laws of the equation (6) are associtaed with the Lie symmetries of the equation in the sense of (11), then conservation laws \( D_i T^i = 0 \) of (6) can be reduced \( \tilde{D}_i \tilde{T}^i = 0 \) under the similarity transformations corresponding to \( \Gamma \) Lie symmetries [25].

Therefore, generalization can be clearly made. If \( s \) th order equation (cf. Eq. (6)) has a non-trivial conserved form and this conserved form is asociated with Lie symmetries (for the \( m \) number of reductions where \( m \) is the number of independent variables of (6)) then the equation can be reduced to a \((s-1)\) th ordinary differential equation (ODE) [37].

### 3. Recapitulation of auxiliary equation method

The principal steps of auxiliary equation method are summarized in this section [38]. Suppose that a nonlinear evolution equation is expressed as
\[
R(u, u_x, u_t, u_{xx}, u_{tt}, \ldots) = 0,
\]
where \( R \) is a polynomial in \( u(x, t) \) and its partial derivatives involve the highest order derivatives and nonlinear terms. After algebraic operations, Eq. (16) is transformed into
an ODE with the transformation $\xi = x - \mu t$

$$\mathcal{D}(u, u', u'', \ldots) = 0. \quad (17)$$

Suppose that the solution of Eq. (17) has the form

$$u(\xi) = S(\phi) = \sum_{j=0}^{M} n_j \phi(\zeta)^j, \quad (18)$$

where the integer $M$ can be obtained by balancing procedure appearing in Eq. (17) and $n_j (j = 0, 1, \ldots, M)$ are constants that need to be determined. Here, $\phi(\zeta)$ fulfills the following auxiliary ODE:

$$\left(\frac{d\phi}{d\zeta}\right)^2 = m_1 \phi(\zeta)^2 + m_2 \phi(\zeta)^4 + m_3 \phi(\zeta)^6, \quad (19)$$

where $m_1, m_2, m_3$ are real parameters. Eq. (19) admits several types of solutions:

**Case 1.** For $m_1 > 0$, $\phi_1(\zeta) = \sqrt{-\frac{m_1 m_2 \text{sech}\left(\sqrt{m_1}\zeta\right)^2}{m_2^2 - m_1 m_3 (1 + \epsilon \tanh(\sqrt{m_1}\zeta))^2}},$

**Case 2.** For $m_1 > 0$, $\phi_2(\zeta) = \sqrt{-\frac{m_1 m_2 \text{csch}\left(\sqrt{m_1}\zeta\right)^2}{m_2^2 - m_1 m_3 (1 + \epsilon \coth(\sqrt{m_1}\zeta))^2}},$

**Case 3.** For $m_1 > 0$, $\Delta > 0$, $\phi_3(\zeta) = \sqrt{2} \frac{m_1}{\epsilon \Delta \cosh(2 \sqrt{m_1}\zeta) - m_2},$

**Case 4.** For $m_1 < 0$, $\Delta > 0$, $\phi_4(\zeta) = \sqrt{2} \frac{m_1}{\epsilon \Delta \cos(2 \sqrt{-m_1}\zeta) - m_2},$

**Case 5.** For $m_1 > 0$, $\Delta < 0$, $\phi_5(\zeta) = \sqrt{2} \frac{m_1}{\epsilon \Delta \cos(2 \sqrt{-m_1}\zeta) - m_2},$

**Case 6.** For $m_1 < 0$, $m_3 > 0$, $\phi_6(\zeta) = \sqrt{2} \frac{m_1}{\epsilon \Delta \sin(2 \sqrt{-m_1}\zeta) - m_2},$

**Case 7.** For $m_1 > 0$, $m_3 > 0$, $\phi_7(\zeta) = \sqrt{-\frac{m_1 \text{sech}\left(\sqrt{m_1}\zeta\right)^2}{m_2^2 - 2 \epsilon \sqrt{m_1} m_3 \tanh(\sqrt{m_1}\zeta)}},$

**Case 8.** For $m_1 < 0$, $m_3 > 0$, $\phi_8(\zeta) = \sqrt{-\frac{m_1 \text{sec}\left(\sqrt{-m_1}\zeta\right)^2}{m_2^2 + 2 \epsilon \sqrt{-m_1} m_3 \tan(\sqrt{-m_1}\zeta)}},$

**Case 9.** For $m_1 > 0$, $m_3 > 0$, $\phi_9(\zeta) = \sqrt{-\frac{m_1 \text{csch}\left(\sqrt{m_1}\zeta\right)^2}{m_2^2 + 2 \epsilon \sqrt{m_1} m_3 \coth(\sqrt{m_1}\zeta)}},$
**Case 10.** For $m_1 < 0$, $m_3 > 0$, \( \phi_{10}(\zeta) = \sqrt{-\frac{m_1 \csc(\sqrt{-m_1} \zeta)}{m_2^2 + 2 \epsilon \sqrt{-m_1 m_3 \tanh(\sqrt{-m_1} \zeta)}}} \).

**Case 11.** For $m_1 > 0$, $\Delta = 0$, \( \phi_{11}(\zeta) = \sqrt{\frac{m_1 (1 + \text{tanh}(1/2 \sqrt{m_1} \zeta))}{m_2}} \).

**Case 12.** For $m_1 > 0$, $\Delta = 0$, \( \phi_{12}(\zeta) = \sqrt{\frac{m_1 (1 + \text{coth}(1/2 \sqrt{m_1} \zeta))}{m_2}} \).

**Case 13.** For $m_1 > 0$, \( \phi_{13}(\zeta) = 4 \sqrt{\frac{m_1 e^{2 \epsilon \sqrt{m_1} \zeta}}{(e^{2 \epsilon \sqrt{m_1} \zeta} - 4 m_2)^2 - 64 m_1 m_3}}. \)

**Case 14.** For $m_1 > 0$, $m_2 = 0$, \( \phi_{14}(\zeta) = 4 \sqrt{\frac{m_1 e^{2 \epsilon \sqrt{m_1} \zeta}}{1 - 64 m_1 m_3 e^{4 \epsilon \sqrt{m_1} \zeta}}}}. \)

where $\Delta = m_2^2 - 4m_1m_3$ and $\epsilon = \pm 1$.

4. **Solutions of the Benjamin-Ono equation**

The symmetry group of the Benjamin-Ono equation (1) will be generated by the vector field of the form

\[
\Gamma = \tau(x, t, u) \frac{\partial}{\partial t} + \zeta(x, t, u) \frac{\partial}{\partial x} + \eta(x, t, u) \frac{\partial}{\partial u}. \tag{20}
\]

We obtain an overdetermined system of linear PDEs implementing the fourth prolongation $\Gamma^{[4]}$ to Eq. (1). Then, solving the obtained system, we get Lie point symmetries of (1) with the help of SADE (in Maple) [39]:

\[
\Gamma_1 = \frac{\partial}{\partial x}, \quad \Gamma_2 = \frac{\partial}{\partial t}, \\
\Gamma_3 = u \frac{\partial}{\partial u} - t \frac{\partial}{\partial t} - \frac{x}{2} \frac{\partial}{\partial x}. \tag{21}
\]

It is shown by Kaplan et al. [40] that (1) accepts the following conserved vectors

\[
T_1^t = xu_t, \quad T_2^t = -u + tu_t, \\
T_1^x = 2\beta xuu_x - \beta u^2 - \gamma uu_{xx} + \gamma xu_{xxx}, \\
T_2^x = 2\beta tuu_x + \gamma tu_{xxx}, \\
T_3^t = -xu + xt u_t, \\
T_3^x = 2\beta txu_x u - \beta t u^2 - \gamma tu_{xx} + \gamma tu_{xxx}, \\
T_4^t = u_t.
\]
\[ T_4^x = 2\beta u_x + \gamma u_{xxx}, \]

with the corresponding multipliers

\[ (\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (x, t, xt, 1). \quad (23) \]

With the help of the double reduction theory, we compute the exact solutions of Eq. (1). If the following expression is satisfied

\[ T^* = \Gamma \left( \frac{T_t^t}{T_x^t} \right) - \left( \frac{D_t \xi^t}{D_x \xi^x} \right) \left( \frac{T_t^t}{T_x^t} \right) + (D_t \xi^t + D_x \xi^x) \left( \frac{T_t^t}{T_x^t} \right) = 0, \quad (24) \]

then the Lie-Bäcklund symmetry generator \( \Gamma \) is associated with a conserved vector \( T \) of Eq. (1).

4.1 A double reduction of (1) by \( \langle \Gamma_1, \Gamma_2 \rangle \)

We now show that \( \Gamma_1 \) and \( \Gamma_2 \) are associated with \( T_4 \). We obtain

\[ \left( \begin{array}{c} T_4^{*t} \\ T_4^{*x} \end{array} \right) = \Gamma_1^{[3]} \left( \begin{array}{c} T_4^t \\ T_4^x \end{array} \right) - \left( \begin{array}{c} 0 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{c} T_4^t \\ T_4^x \end{array} \right) + (0) \left( \begin{array}{c} T_4^t \\ T_4^x \end{array} \right), \quad (25) \]

from (24). Here \( \Gamma_1^{[3]} = \frac{\partial}{\partial x} \). (25) shows that

\[ T_4^{*t} = 0, T_4^{*x} = 0. \]

Thus, \( \Gamma_1 \) is associated with \( T_4 \) [35].

Similarly for \( \Gamma_2 \), we obtain

\[ \left( \begin{array}{c} T_4^{*t} \\ T_4^{*x} \end{array} \right) = \Gamma_2^{[3]} \left( \begin{array}{c} T_4^t \\ T_4^x \end{array} \right) - \left( \begin{array}{c} 0 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{c} T_4^t \\ T_4^x \end{array} \right) + (0) \left( \begin{array}{c} T_4^t \\ T_4^x \end{array} \right), \quad (26) \]

where \( \Gamma_2^{[3]} = \frac{\partial}{\partial t} \). (26) shows that

\[ T_4^{*t} = 0, T_4^{*x} = 0. \]

Thus, \( \Gamma_2 \) is associated with \( T_4 \) in the sense of Kara and Mahomed’s definition [35].

We investigate a linear combination of \( \Gamma_1 \) and \( \Gamma_2 \):

\[ \Gamma = \alpha \Gamma_1 + \Gamma_2 = \alpha \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad (27) \]

and this generator is then transformed into its canonical form \( Y = \frac{\partial}{\partial s} \) where we suppose that \( Y \) has the following form

\[ Y = \frac{\partial}{\partial s} + 0 \frac{\partial}{\partial r} + 0 \frac{\partial}{\partial \omega}. \quad (28) \]

Thus, we get
\[
\frac{dx}{\alpha} = \frac{dt}{1} = \frac{du}{0} = \frac{ds}{1} = \frac{dr}{0} = \frac{d\omega}{0}. \tag{29}
\]

The invariants of (27) from (29) are given by

\[
\begin{align*}
\frac{dt}{1} &= \frac{dx}{\alpha}, \\
\frac{ds}{1} &= \frac{dr}{0}, \quad \frac{d\omega}{0}, \quad \frac{du}{0}, \quad \frac{dr}{0}, \quad \frac{d\omega}{0}, \\
\end{align*} \tag{30}
\]

and

\[
b_1 = \alpha t - x, \quad b_2 = s - t, \quad b_3 = r, \quad b_4 = \omega, \quad b_5 = u. \tag{31}
\]

By choosing \(b_1 = b_3, \quad b_2 = 0, \quad b_4 = b_5,\) we obtain the canonical coordinates

\[
r = \alpha t - x, s = t, \quad \omega = u, \tag{32}
\]

where \(w = w(r).\) The inverse canonical coordinates are presented below

\[
x = \alpha s - r, \quad t = s, \quad u = \omega. \tag{33}
\]

The matrices \(A\) and \(A^{-1}\) can be computed using the canonical coordinates above

\[
A = \begin{pmatrix}
D_r t & D_r x \\
D_s t & D_s x
\end{pmatrix} = \begin{pmatrix}
0 & -1 \\
1 & \alpha
\end{pmatrix}
\]

and

\[
(A^{-1})^T = \begin{pmatrix}
D_r r & D_r x \\
D_s r & D_s x
\end{pmatrix} = \begin{pmatrix}
\alpha & -1 \\
1 & 0
\end{pmatrix}
\]

where \(J = det A = 1.\) The reduced conserved form is given by

\[
\begin{pmatrix}
T_4^r \\
T_4^s
\end{pmatrix} = J(A^{-1})^T \begin{pmatrix}
T_4^r \\
T_4^s
\end{pmatrix} = \begin{pmatrix}
\alpha & -1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
T_4^r \\
T_4^s
\end{pmatrix} = \begin{pmatrix}
\alpha T_4^r - T_4^x \\
T_4^x
\end{pmatrix}. \tag{34}
\]

Substituting (32) and the partial derivatives of \(u,\) into (34), we get

\[
\begin{align*}
T_4^r &= \alpha^2 \omega_r + 2\beta w w_r + \gamma \omega_{rrr}, \\
T_4^s &= -2\beta w w_r - \gamma \omega_{rrr},
\end{align*} \tag{35}
\]

where the reduced conserved form (35) satisfies

\[
D_r T_4^r + D_s T_4^s = 0. \tag{36}
\]

The reduced form (36) also satisfies \(D_r T_4^r = 0.\) This yields

\[
\alpha^2 \omega_r + 2\beta w w_r + \gamma \omega_{rrr} = k_1. \tag{37}
\]
4.2. Application of the auxiliary equation method

We seek solutions of (37) by the auxiliary equation method while setting the constant \( k_1 \) to zero. Balancing \( w_{rrr} \) and \( ww_r \) in Eq.(37), we have \( M = 2 \) and then proceeded as

\[
w(r) = n_0 + n_1 \phi(r) + n_2 \phi(r)^2, \tag{38}\]

with \( n_0, n_1 \) and \( n_2 \) which are constants need to be determined. Substituting Eq. (38) into Eq. (37) and equating the coefficients \( \phi^j(r) \) for \( (j = 0, 1, 2, \ldots, M) \) to zero, a system of algebraic equations was obtained. We recovered solutions for the obtained system as

\[
\{m_2 = - \frac{\beta n_2}{6\gamma}, m_3 = 0, n_0 = - \frac{4\gamma m_1 + \alpha^2}{2\beta}, n_1 = 0, \}\]  \tag{39}

With the help of inverse canonical coordinates, the solutions of Eq. (1) are obtained as follows when \( m_1 > 0 \)

\[
u_1(x, t) = - \frac{\alpha^2 + 4\gamma m_1}{2\beta} + 6 \frac{m_1 \gamma \left( sech(\sqrt{m_1}(\alpha t-x)) \right)^2}{\beta}, \tag{40}\]

\[
u_2(x, t) = - \frac{\alpha^2 + 4\gamma m_1}{2\beta} - 6 \frac{m_1 \gamma \left( csch(\sqrt{m_1}(\alpha t-x)) \right)^2}{\beta}, \tag{41}\]

\[
u_3(x, t) = - \frac{\alpha^2 + 4\gamma m_1}{2\beta} + 2n_2 m_1 \left( \frac{e^{\sqrt{36} \frac{\beta^2 n_2^2}{\gamma^2}} \cos(2 \sqrt{m_1}(\alpha t-x)) + \frac{\beta n_2}{6\gamma}}{\sqrt{36}} \right)^{-1}, \tag{42}\]

\[
u_4(x, t) = - \frac{\alpha^2 + 4\gamma m_1}{2\beta} + 16n_2 m_1 e^{2\sqrt{36}(\alpha t-x)} \left( e^{2\sqrt{36}(\alpha t-x)} \right)^2 + 2 \frac{\beta n_2}{3\gamma} \right)^{-2}, \tag{43}\]

and when \( m_1 < 0 \)

\[
u_5(x, t) = - \frac{\alpha^2 + 4\gamma m_1}{2\beta} + 2n_2 m_1 \left( \frac{e^{\sqrt{36} \frac{\beta^2 n_2^2}{\gamma^2}} \cos(2 \sqrt{-m_1}(\alpha t-x)) + \frac{\beta n_2}{6\gamma}}{\sqrt{36}} \right)^{-1}, \tag{44}\]

\[
u_6(x, t) = - \frac{\alpha^2 + 4\gamma m_1}{2\beta} + 2n_2 m_1 \left( \frac{e^{\sqrt{36} \frac{\beta^2 n_2^2}{\gamma^2}} \sin(2 \sqrt{-m_1}(\alpha t-x)) + \frac{\beta n_2}{6\gamma}}{\sqrt{36}} \right)^{-1}. \tag{45}\]

Figure 1. Profile of solution (40) where \( \gamma = 1, \beta = 1.2 \) with (a) \( \alpha = 0.5, m_1 = 0.1 \), (b) \( \alpha = 0.9, m_1 = 0.5 \) and (c) \( \alpha = 1.2, m_1 = 1.5 \).
Figure 2. Profile of solution (40) where \( \gamma = 1, \beta = 2 \) with (d) \( \alpha = 1, m_1 = 2.5 \), (e) \( \alpha = -1.1, m_1 = 5 \) and (f) \( \alpha = 1.1, m_1 = 4 \).

Figure 3. Profile of solution (43) where \( \gamma = 2, \beta = -2, \epsilon = -1, n_2 = 1.2 \) with \{\( \alpha = 2.5, m_1 = 0.4 \}\}, \{\( \alpha = 2.2, m_1 = 1 \}\} and \{\( \alpha = 1.8, m_1 = 2.1 \}\}, respectively.

Figure 4. Profile of solution (44) where \( \gamma = 1, \beta = 1, \epsilon = 1, n_2 = 0.6 \) with \{\( \alpha = -2, m_1 = -0.2 \}\}, \{\( \alpha = 1.5, m_1 = -0.5 \}\} and \{\( \alpha = 1.5, m_1 = -1 \}\}, respectively.
Figure 5. Profile of solution (44) where $\gamma = 1, \beta = 1, \epsilon = 1, n_2 = 0.6, \alpha = 1$ with $m_1 = -1, m_1 = -1.5$ and $m_1 = -2$, respectively.

Remark 1 The accuracy of all the solutions obtained was examined by placing them in their original equations using Maple.

6. Discussions

This work provides a new way of constructing various exact solutions for PDEs by establishing a relationship of the current symmetry with the conserved vectors of the equation. In order to find the solutions of the reduced equation obtained as a result of double reduction theory which has been applied after establishing the conserved vectors association with the Lie symmetries, the auxiliary equation method, which is an effective method, was used. We have achieved various traveling wave solutions including trigonometric, hyperbolic, and exponential solutions. In Figs. 1-5, a few graphic representations are given by giving special values to the parameters in the solutions obtained and the behavior caused by small changes in parameters is shown in 3D graphics. Figs. 1 and 2 represent solitary wave solution, Fig. 3 represent hyperbolic function solution, and Figs. 4-5 demonstrate periodic wave solutions which are traveling wave solutions that repeat its values in regular intervals or periods.

5. Concluding remarks

In this work, we considered Benjamin-Ono equation and we used the double reduction theory and the auxiliary equation method to investigate underlying equation. Double reduction theory is a powerful mathematical tool for obtaining reduced forms and exact solutions of partial differential equations or systems. This theory provides not only transformations that provide traveling wave solutions, but also a systematic way of finding other types of transformations. These transformations reduce a nonlinear system of $q$th-order PDEs with $n$ independent and $m$ dependent variables to a nonlinear system of $(q - 1)$th-order ODEs provided that in every reduction at least one symmetry is associated with a nontrivial conserved vector; otherwise, reduction is not possible. The reduced ODE can be solved analytically or numerically to obtain exact or approximate solutions. Interestingly, the transformations that give traveling wave solutions can sometimes give more than one reduced form and the simple one can be used to find the exact solution [41]. Using the association between Lie point symmetry generators and conservation law of Benjamin-Ono equation, we obtained a reduction in the number of both orders and variables of the underlying equation. Therefore, we reduced the number
of variables from two to one and the order of the equation from four to three, at the same
time. The application of the double reduction method, which is the main purpose of this
study, has been successfully completed. To obtain solutions of the reduced equation, we
have successfully applied auxiliary equation method. These solutions include a periodic,
parabolic, and exponential solutions confirming the effectiveness of the method. When
some of the solutions obtained are compared with the studies in the literature and when
parameters are given arbitrary values in works in which used different methods, it can be
observed that solutions with similar form are obtained [29]. According to our best
knowledge, the remaining solutions are new. The physical properties of some obtained
results have been illustrated using suitable parameter values in Figure. 1 – 5. We hope
that the results obtained will be used for important physical practices and guide new
research.

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