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Conchoid curves and surfaces in Euclidean 3-Space

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Abstract

In this study firstly, we study with conchoid curves in Euclidean plane E^2. We calculate the curvature of the conchoid curve and give some results. Furthermore, we consider the surface of revolution given with the conchoid curve in Euclidean 3-space E^3. The Gaussian and mean curvature is calculated of these surfaces. Also we give some examples and plot their graphics. Finally we study conchoidal surface in Euclidean 3-space. We give some results for the conchoidal surface to become flat and minimal. We give an example and plot the garphics of the conchoidal surfaces.

Keywords: *Conchoid, Limaçons Pascal, Gaussian curvature, mean curvature.*

3-boyutlu Öklid uzayında Conchoid eğri ve yüzeyleri

Özet

Bu çalışmada ilk olarak düzlemde conchoid eğrileri çalışılmıştır. Conchoid eğrisinin eğriliği hesaplanıp bazı sonuçlar verilmiştir. Ayrıca 3-boyutlu Öklid uzayında conchoid eğrisiyle elde edilen dönel yüzeyler ele alınmıştır. Bu yüzeylerin Gauss ve ortalama eğrilikleri hesaplanmış, bunlarla ilgili örnekler verilip grafikleri çizdirilmiştir. Son olarak 3-boyutlu Öklid uzayında conchoidal yüzeyler üzerinde durulmuş ve conchoidal yüzeylerin flat ve minimal olma şartlarına bakılmıştır. Conchoidal yüzey örnekleri de verilip grafikleri çizdirilmiştir.

Anahtar kelimeler: *Conchoid, Pascal Limaçonu, Gauss eğrilik, ortalama eğrilik*

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1. Introduction

The invention of the plane curve conchoid (`mussel-shell shaped') by the Greek mathematician Nicomedes, who applied it to the problem of the duplication of the cube and of trisecting an angle. It was a favorite with the mathematicians of the seventeenth century [10].

The well-known construction of conchoids is usually applied to curves in the Euclidean plane E^2 [1]. The conchoid transformation has been applied to surfaces in Euclidean three-space E^3 in ([6], [11], [13], [14], [15]) in order to construct new classes of surfaces admitting rational parametrizations, and thus, making them accessible to the algorithms implemented in CAD systems. Algebraic attributes of conchoid curves and surfaces have been studied in [16], [17]. Also the spacelike conchoid curves in the Minkowski plane was studied in [3].

In this paper in the Section 2 we give some preliminaries of the curves and surfaces in E^3. Section 3 tells about the planar conchoid curves and their curvatures. In Section 4 we consider surface of revolution whose rotating curve is a conchoid and we obtain Gaussian and mean curvature. In the final section we consider conchoidal surface in Euclidean 3-space. We give some results for the conchoidal surfaces to become a flat and minimal. Finally we give some examples and plot their graphics.

2. Basic concepts

We now recall some basic concepts of the curves and surfaces in E^3.

2.1. Curves in E^3

Let $\alpha : I \subset R \to E^3$ be a regular curve. For the Frenet frame $\{T, N, B\}$ of α the Frenet-Serret formulas hold;

$$
\begin{aligned}
&T'(s) = v(s)\kappa(s)N(s),\\
&N'(s) = v(s)(-\kappa(s)T(s) + \tau(s)B(s)),\\
&B'(s) = -v(s)\tau(s)N(s)
\end{aligned}
$$

where $v(s) = \|\alpha'(s)\|$ is the speed function of α and $\kappa(s)$ and $\tau(s)$ are Frenet curvatures defined by:

$$\kappa(s) = \frac{\|\alpha'(s) \times \alpha''(s)\|}{\|\alpha'(s)\|^3}, \tag{2.1}$$

and

$$\tau(s) = \frac{\langle \alpha'(s) \times \alpha''(s), \alpha'''(s)\rangle}{\|\alpha'(s) \times \alpha''(s)\|^2} \tag{2.2}$$

respectively (see, [5], [12]).

2.1. Surfaces in E^3

Let M be a smooth surface in E^3 given with the patch $X(u,v):(u,v)\in D\subset E^2$. The tangent space to M at an arbitrary point $p=X(u,v)$ of M $span\{X_u, X_v\}$. Let N be the unit normal vector field defined by $N=\dfrac{X_u\times X_v}{\left\|X_u\times X_v\right\|}$.

Then the coefficients of the first and second fundamental forms of the surface M are defined respectively as

$$\begin{aligned} E &= \langle X_u, X_u\rangle, \\ F &= \langle X_u, X_v\rangle, \\ G &= \langle X_v, X_v\rangle \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} e &= \langle X_{uu}, N\rangle, \\ f &= \langle X_{uv}, N\rangle, \\ g &= \langle X_{vv}, N\rangle \end{aligned} \tag{2.4}$$

where $\langle , \rangle$ is the Euclidean inner product. The surface patch is regular, i.e.,$W^2=EG-F^2\neq 0$. Further, the Gaussian curvature and mean curvature of the surface are given by

$$K=\frac{eg-f^2}{EG-F^2} \tag{2.5}$$

and

$$H=\frac{eG+gE-2fF}{2(EG-F^2)} \tag{2.6}$$

respectively.

The surface is called flat and minimal if its Gaussian curvature and mean curvature vanishes respectively ([5], [12]).

3. Conchoid curves in E^2

Given a planar curve *c*, a fixed point *A* in the plane, and constant distance *d*. The conchoid to *c* from the focus *A* at distance *d* is the set of points *Q* in the line *A*P at distance *d* of a point *P* varying in the curve *c*. The well known two classical conchoids are the conchoids of Nicomedes (planar curve is a line) and Limaçons of Pascal (planar curve is a circle) [16]. Conchoids are useful in many applications as conic reflection and refraction in physics and optics, electrode of static field, fluid processing in mechanics, etc. (see, [2], [7], [8], [9], [18], [19]).

In this section we consider conchoid curves in Euclidean plane E^2. We calculate the curvature of the curve c and its conchoid curve d. We give some examples and plot their graphics.

Definition 1. [14] Let $c: I \subset R \to E^2$ be Euclidean plane curve and its polar representation is $c(t) = r(t)(\cos t, \sin t)$. Its conchoid curve D with respect O and distance d is defined by $d(t) = (r(t) \pm d)(\cos t, \sin t)$. We can consider any parametrization $k(t)$ of the unit circle S^1. The curve C and its conchoid curves D are represented by

$$c(t) = r(t)k(t) \tag{3.1}$$

and

$$c(t) = (r(t) \pm d)k(t) \tag{3.2}$$

where $\|k(t)\| = 1$.

In the following results we give the curvature of the planar curve C and its conchoid curve D.

Proposition 1. Let $c: I \subset R \to E^2$ be planar curve given with the polar representation (3.1). Then the curvature $\kappa(t)$ of $c(t)$ becomes

$$\kappa(t) = \frac{2(r')^2 - rr'' + r^2}{(r^2 + (r')^2)^{3/2}}.$$

Proof. Using the equation (3.1) we obtain the first and second derivatives of the curve c
$c'(t) = (r'\cos t - r\sin t, r'\sin t + r\cos t),.$
$c''(t) = (r''\cos t - 2r'\sin t - r\cos t, r''\sin t + 2r'\cos t - r\sin t)$.
Substituting this derivatives into (2.1) we get the result.

Proposition 2. Let $d: I \subset R \to E^2$ be conchoid curve of c given with the polar representation (3.2). Then the curvature $\kappa_d(t)$ of $d(t)$ becomes

$$\kappa_d(t) = \frac{2(r')^2 - (r \pm d)r'' + (r \pm d)^2}{((r \pm d)^2 + (r')^2)^{3/2}}.$$

Corollary 1. Let $c: I \subset R \to E^2$ be planar curve given with the polar representation (3.1). If *c* is a straight line then $r(t) = \dfrac{1}{c_1 \sin t - c_2 \cos t}$.

Proof. Let $c: I \subset R \to E^2$ be planar curve given with the polar representation (3.1). Assume that c is a straight line then $\kappa(t)=0$. So we get $2(r')^2 - rr'' + r^2 = 0$ and solving this differential equation we obtain the result.

Corollary 2. Let $c: I \subset R \to E^2$ be planar curve given with the polar representation (3.1). If c is a unit speed curve then c is a circle with center $\left(\frac{c_1}{2}, \frac{c_2}{2}\right)$ where c_1, c_2 are real constant satisfying the condition $c_1^2 + c_2^2 = 1$.

Proof. Let $c: I \subset R \to E^2$ be planar curve given with the polar representation (3.1). Assume that c is a unit speed curve then the norm of the derivative of the curve $\|c'(t)\| = r^2 + (r')^2 = 1$. So, solving this differential equation we get $r(t) = c_1 \cos t + c_2 \sin t$ where c_1, c_2 are real constant satisfying the condition $c_1^2 + c_2^2 = 1$. Furthermore the polar representation of the curve is a circle with the center $\left(\frac{c_1}{2}, \frac{c_2}{2}\right)$.

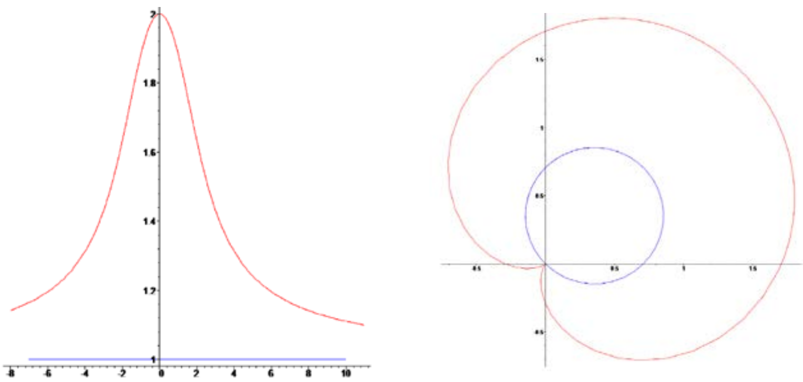
We give a result of [4];

Theorem 1. Pascal's limaçon is a conchoid of a circle.

We give the following examples;

Example 1. 1) Let c be a straight line then $c(t) = \frac{1}{\sin t}(\cos t, \sin t)$ and its conchoid curve $c_d(t) = (\frac{1}{\sin t} \pm d)(\cos t, \sin t)$. (the curve c is blue and the curve c_d is red) (conchoid of Nicomedes), (Figure 1a).

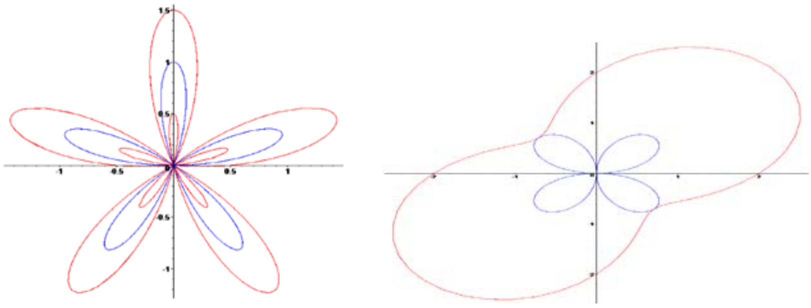
2) Let c be a circle then $c(t) = \frac{1}{\sqrt{2}}(\cos t + \sin t)(\cos t, \sin t)$ and its conchoid curve $c_d(t) = \left(\frac{1}{\sqrt{2}}(\cos t + \sin t) \pm d\right)(\cos t, \sin t)$ (Pascal Limaçon), (Figure 1b)



a) $d=1$ b) $d=-1$

Figure 1. Line and circle and its conchoids.

3) Let the function $r(t) = \sin at$, $a \in R$ then $c(t) = \sin at(\cos t, \sin t)$ and its conchoid curve $c_d(t) = (\sin at \pm d)(\cos t, \sin t)$ (the curve c is blue and the curve c_d is red) (rose curve and botanical curve), (Figure 2a,b)



a) $a = 5, d = \frac{1}{2}$ b) $a = 2, d = 2$

Figure 2. Botanical curves and conchoid curves.

4. Surface of revolution given with Conchoid curves in E^3

In this section we consider surface of revolution with the rotating curve $c(t)$ and its conchoid curve $c_d(t)$. We obtain the Gaussian and mean curvature of the surfaces and give some examples.

Let M be a surface of revolution generated by curve $c(t)$ given with (3.1). Consequently, the surface given with the surface patch

$$X(t,s) = (r(t)\cos t, r(t)\sin t\cos s, r(t)\sin t\sin s) \tag{4.1}$$

Let M_d be a surface of revolution generated by conchoid curve $c_d(t)$ given with (3.2). Consequently, the surface parametrized by

$$\tilde{X}(t,s) = ((r(t) \pm d)\cos t, (r(t) \pm d)\sin t\cos s, (r(t) \pm d)\sin t\sin s) \tag{4.2}$$

Theorem 2. Let M be a surface of revolution given with the patch (4.1). Then the Gaussian curvature K of M becomes

$$K = \frac{(r'\cos t - r\sin t)(rr'' - 2(r')^2 - r^2)}{r\sin t(r^2 + (r')^2)^2} \tag{4.3}$$

Proof. The surface M is spanned by the vector fields

$$\frac{\partial X}{\partial t} = (r'\cos t - r\sin t, (r'\sin t + r\cos t)\cos s, (r'\sin t + r\cos t)\sin s),$$
$$\frac{\partial X}{\partial s} = (0, -r\sin t\sin s, r\sin t\cos s)$$

Hence the coefficients of the first fundamental form are

$$\begin{aligned} E &= \langle X_t, X_t \rangle = r^2 + (r')^2, \\ F &= \langle X_t, X_s \rangle = 0, \\ G &= \langle X_s, X_s \rangle = r^2 \sin^2 t \end{aligned} \tag{4.4}$$

The second partial derivatives of $X(t,s)$ are expressed as follows

$$\begin{aligned} X_{tt} &= ((r''-r)\cos t - 2r'\sin t, ((r''-r)\sin t + 2r'\cos t)\cos s, ((r''-r)\sin t + 2r'\cos t)\sin s), \\ X_{ts} &= (0, -(r'\sin t + r\cos t)\sin s, (r'\sin t + r\cos t)\cos s), \\ X_{ss} &= (0, -r\sin t\cos s, -r\sin t\sin s), \end{aligned} \tag{4.5}$$

Further, the unit normal vector of M is

$$N = \frac{1}{\sqrt{r^2+(r')^2}}(r'\sin t + r\cos t, (r\sin t - r'\cos t)\cos s, (r\sin t - r'\cos t)\sin s) \tag{4.6}$$

Using (2.4), (4.5) and (4.6) we obtain the coefficients of the second fundamental form,

$$\begin{aligned} e &= \frac{rr'' - 2(r')^2 - r^2}{\sqrt{r^2+(r')^2}}, \\ f &= 0, \\ g &= \frac{r\sin t(r'\cos t - r\sin t)}{\sqrt{r^2+(r')^2}} \end{aligned} \tag{4.7}$$

Further, substituting (4.4) and (4.7) into (2.5) we get (4.3).

Theorem 3. Let M be a surface of revolution given with the patch (4.1). Then the mean curvature of M becomes

$$H = \frac{r\sin t(rr'' - 2(r')^2 - r^2) + (r^2+(r')^2)(r'\cos t - r\sin t)}{2r\sin t(r^2+(r')^2)^{3/2}} \tag{4.8}$$

Proof. Using the equations (2.6), (4.4) and (4.7) we get the result.

As a result of Theorem 2 we obtain the following corollaries.

Corollary 3. Let M be a surface of revolution given with the patch (4.1). If

$r(t) = \dfrac{c_1}{\cos t}$ or $r(t) = \dfrac{c_1}{c_2\sin t - c_3\cos t}$

then M is a flat surface which is a part of plane, cylinder or cone, where c_1, c_2, c_3 are real constants.

Corollary 4. Let *M* be a surface of revolution given with the patch (4.1). If $r(t)=\frac{c_1}{\cos t}$ then *M* is a minimal surface which is a part of a plane, where c_1 is real constant.

Using the similar way one can give these results for surface of revolution given with the conchoid curves.

Theorem 4. Let M_d be a surface of revolution given with the patch (4.2). Then the Gaussian curvature K_d of M_d becomes

$$K_d=\frac{(r'\cos t-(r(t)\pm d)\sin t)((r(t)\pm d)r''-2(r')^2-(r(t)\pm d)^2)}{(r(t)\pm d)\sin t((r(t)\pm d)^2+(r')^2)^2} \tag{4.9}$$

Theorem 5. Let M_d be a surface of revolution given with the patch (4.2). Then the mean curvature of M_d becomes

$$H_d=\frac{(r\pm d)\sin t((r\pm d)r''-2(r')^2-(r\pm d)^2)+((r\pm d)^2+(r')^2)(r'\cos t-(r\pm d)\sin t)}{2(r\pm d)\sin t((r\pm d)^2+(r')^2)^{3/2}} \tag{4.10}$$

As a consequence of Theorem 4 we obtain the following results.

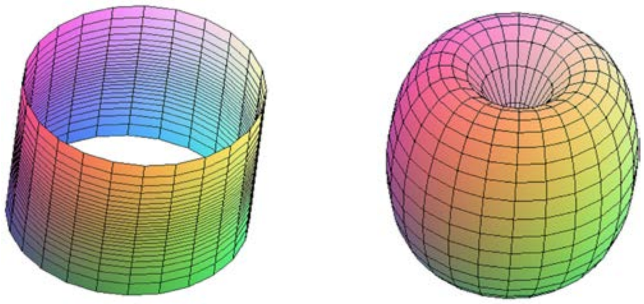
Corollary 5. Let M_d be a surface of revolution given with the patch (4.2). If

$$r(t)=\pm d+\frac{c_1}{\cos t} \quad \text{or} \quad r(t)=\pm d+\frac{c_1}{c_2\sin t-c_3\cos t}$$

then M_d is a flat surface, where c_1, c_2, c_3 are real constants.

Corollary 6. Let M_d be a surface of revolution given with the patch (4.2). If $r(t)=\pm d+\frac{c_1}{\cos t}$ then M_d is a minimal surface, where c_1 is real constant.
We give some examples;

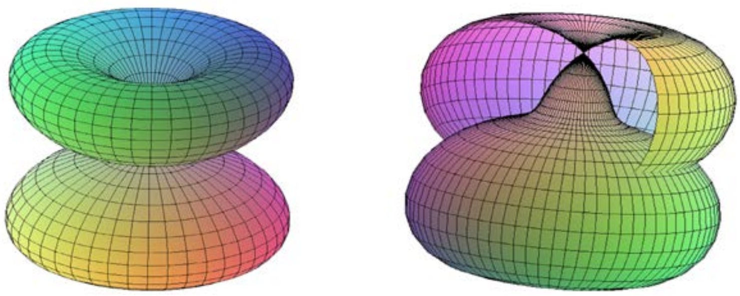
Example 2. 1) Let the rotating curve c be a straight line then the surface of revolution *M* becomes a flat surface given with the parametrization $X(t,s)=\frac{1}{\sin t}(\cos t,\sin t\cos s,\sin t\sin s)$, (Figure 3a). Further for $d=-2$ the surface of revolution M_d has the form $\tilde{X}(t,s)=(\frac{1}{\sin t}-2)(\cos t,\sin t\cos s,\sin t\sin s)$, (Figure 3b).



a) b)

Figure 3. Flat rotational surface and its conchoid.

2) Let the rotating curve $c(t) = (\sin 2t \cos t, \sin 2t \sin t)$ so the the surface of revolution parametrized by $X(t,s) = (\sin 2t \cos t, \sin 2t \sin t \cos s, \sin 2t \sin t \sin s)$, (Figure 4a). Further for $d = 2$ the surface of revolution M_d has the form $\tilde{X}(t,s) = (\sin 2t + 2)(\cos t, \sin t \cos s, \sin t \sin s)$, (Figure 4b).



a) b)

Figure 4. Surface of revolution and its conchoid with $r(t) = \sin 2t$.

5. Conchoidal surfaces in E^3

The conchoidal surface F_d of a given surface F is obtained by increasing the radius function by d with respect to a given reference point O. Consider $F \subset R^3$ be a regular surface, distance $d \in R$, with respect to a given fixed point $O = (0,0,0) \subset R^3$. Let F be represented by polar representation

$$f(u,v) = r(u,v)\rho(u,v) \tag{5.1}$$

with $\|\rho(u,v)\| = 1$.

Taking into account the parametrization $\rho(u,v) = (\cos u \cos v, \sin u \cos v, \sin v)$ of the unit sphere S^2, so $\rho(u,v)$ is called spherical part of $f(u,v)$ and $r(u,v)$ its radius function. The conchoidal surface F_d of F at distance d parameterized by

$$f_d(u,v) = (r(u,v) \pm d)\rho(u,v) \tag{5.2}$$

(see,[14]).

Theorem 6. Let F be a regular surface given with the parametrization (5.1). Then the Gaussian curvature F becomes

$$K=-\frac{1}{r^2((r^2+r_v^2)\cos^2 v+r_u^2)^2}((rr_{uv}\cos v-2r_ur_v\cos v+rr_u\sin v)^2 \\ -\cos^2 v(2r_u^2+rr_v\sin v\cos v+r^2\cos^2 v-rr_{uu})(2r_v^2+r^2-rr_{vv}) \tag{5.3}$$

Proof. The tangent space of F is spanned by the vector fields

$$\frac{\partial f}{\partial u}=(r_u\cos u\cos v-r\sin u\cos v, r_u\sin u\cos v+r\cos u\cos v, r_u\sin v),$$
$$\frac{\partial f}{\partial v}=(r_v\cos u\cos v-r\cos u\sin v, r_v\sin u\cos v-r\sin u\sin v, r_v\sin v+r\cos v).$$

Hence the coefficients of the first fundamental form of the surface are

$$\begin{aligned} E&=\langle f_u, f_u\rangle=r^2\cos^2 v+r_u^2,\\ F&=\langle f_u, f_v\rangle=r_ur_v,\\ G&=\langle f_v, f_v\rangle=r^2+r_v^2. \end{aligned} \tag{5.4}$$

The second partial derivatives of $f(u,v)$ are expressed as follows

$$\begin{aligned} f_{uu}&=((r_{uu}-r)\cos u\cos v-2r_u\sin u\cos v,(r_{uu}-r)\sin u\cos v+2r_u\cos u\cos v, r_{uu}\sin v),\\ f_{uv}&=(r_{uv}\cos u\cos v-r_u\cos u\sin v-r_v\sin u\cos v+r\sin u\sin v,\\ &\quad r_{uv}\sin u\cos v-r_u\sin u\sin v+r_v\cos u\cos v-r\cos u\sin v, r_{uv}\sin v+r_u\cos v),\\ f_{vv}&=((r_{vv}-r)\cos u\cos v-2r_v\cos u\sin v,(r_{vv}-r)\sin u\cos v-2r_v\sin u\sin v,\\ &\quad r_{vv}\sin v+2r_v\cos v-r\sin v). \end{aligned} \tag{5.5}$$

The unit normal vector of $f(u,v)$ is

$$\begin{aligned} N=\frac{1}{\sqrt{(r^2+r_v^2)\cos^2 v+r_u^2}}&(r_v\cos u\cos v\sin v+r\cos u\cos^2 v+r_u\sin u,\\ &r_v\sin u\cos v\sin v+r\sin u\cos^2 v-r_u\cos u,\\ &-r_v\cos^2 v+r\cos v\sin v). \end{aligned} \tag{5.6}$$

Using (2.4), (5.5) and (5.6) we obtain the coefficients of the second fundamental form as follows:

$$
\begin{aligned}
e &= -\frac{\cos v(2r_u^2 + rr_v \sin v \cos v + r^2 \cos^2 v - rr_{uu})}{\sqrt{(r^2 + r_v^2)\cos^2 v + r_u^2}}, \\
f &= -\frac{rr_{uv} \cos v - 2r_u r_v \cos v + rr_u \sin v)}{\sqrt{(r^2 + r_v^2)\cos^2 v + r_u^2}}, \\
g &= -\frac{\cos v(2r_v^2 + r^2 - rr_{vv})}{\sqrt{(r^2 + r_v^2)\cos^2 v + r_u^2}}.
\end{aligned} \tag{5.7}
$$

Further, substituting (5.4) and (5.7) into (2.5) we get (5.3).

Theorem 7. Let F be a regular surface given with the parametrization (5.1). Then the mean curvature of F becomes

$$
\begin{aligned}
H = -\frac{1}{2r^2((r^2 + r_v^2)\cos^2 v + r_u^2)^{3/2}} & (\cos v(2r_u^2 + rr_v \sin v \cos v + r^2 \cos^2 v - rr_{uu})(r^2 + r_v^2) \\
& + \cos v(2r_v^2 + r^2 - rr_{vv})(r^2 \cos^2 v + r_u^2) \\
& + 2r_u r_v (rr_{uv} \cos v - 2r_u r_v \cos v + rr_u \sin v)).
\end{aligned}
$$

Proof. Using the equations (2.6), (5.4) and (5.7) we get the result.

Corollary 7. Let F be a regular surface given with the parametrization (5.1).
i) If the radius function $r(u,v)$ be a u-parameter function then the Gaussian and mean curvature of F

$$
K = \frac{\cos^2 v(2r_u^2 + r^2 \cos^2 v - rr_{uu}) - r_u^2 \sin^2 v}{(r^2 \cos^2 v + r_u^2)^2}
$$

and

$$
H = -\frac{\cos v(3r_u^2 + 2r^2 \cos^2 v - rr_{uu})}{2(r^2 \cos^2 v + r_u^2)^{3/2}}
$$

ii) If the radius function $r(u,v)$ be a v-parameter function then the Gaussian and mean curvature of F

$$
K = \frac{(r_v \sin v + r \cos v)(2r_v^2 + r^2 - rr_{vv})}{r \cos v(r^2 + r_v^2)^2}
$$

and

$$
H = -\frac{(r_v \sin v + r \cos v)(r^2 + r_v^2) + r \cos v(2r_v^2 + r^2 - rr_{vv})}{2r \cos v(r^2 + r_v^2)^{3/2}}
$$

Corollary 8. Let F be a regular surface given with the parametrization (5.1). If u-parameter radius function

$$r(u)=\left(\frac{(2c^2-1)(-c_1c_2\sin(2\sqrt{2c^2-1}u)+\sin^2(\sqrt{2c^2-1}u)(c_1^2-c_2^2)+c_2^2}{c^4}\right)^{\frac{-c^2}{2(2c^2-1)}}$$

then F is flat and if

$$r(u)=\pm\frac{\sqrt{c}}{\sqrt{c_3\sin(2cu)-c_4\cos(2cu)}}$$

then F is minimal where $c=\cos v$ and c_1,c_2,c_3,c_4 are real constants.

Corollary 9. Let F be a regular surface given with the parametrization (5.1).
If v-parameter radius function $r(v)=\dfrac{1}{c_1\sin v-c_2\cos v}$ then F is flat and also if $c_2=0$ then F is minimal.

Using the similar way we obtain the Gaussian and mean curvature of the conchoidal surface F_d with respect to the distance d.

Theorem 8. Let F_d be a conchoidal surface of F given with the parametrization (5.2). Then the Gaussian curvature F_d becomes

$$\begin{aligned}\tilde{K}=-\frac{1}{(r\pm d)^2(((r\pm d)^2+r_v^2)\cos^2 v+r_u^2)^2}&(((r\pm d)r_{uv}\cos v-2r_ur_v\cos v+(r\pm d)r_u\sin v)^2\\&-\cos^2 v(2r_u^2+(r\pm d)r_v\sin v\cos v+(r\pm d)^2\cos^2 v-(r\pm d)r_{uu})(2r_v^2+(r\pm d)^2-(r\pm d)r_{vv}))\end{aligned}$$

Theorem 9. Let F_d be a conchoidal surface of F given with the parametrization (5.2). Then the mean curvature of F_d becomes

$$\begin{aligned}\tilde{H}=-\frac{1}{2(r\pm d)^2(((r\pm d)^2+r_v^2)\cos^2 v+r_u^2)^{3/2}}&(\cos v(2r_u^2+(r\pm d)r_v\sin v\cos v+(r\pm d)^2\cos^2 v-(r\pm d)r_{uu})((r\pm d)^2+r_v^2)\\&+\cos v(2r_v^2+(r\pm d)^2-(r\pm d)r_{vv})((r\pm d)^2\cos^2 v+r_u^2)\\&+2r_ur_v((r\pm d)r_{uv}\cos v-2r_ur_v\cos v+(r\pm d)r_u\sin v))\end{aligned}$$

Corollary 10. Let F_d be a conchoidal surface of F given with the parametrization (5.2).
i) If the radius function $r(u,v)$ be a u-parameter function then the Gaussian and mean curvature of F_d

$$\tilde{K}=\frac{\cos^2 v(2r_u^2+(r\pm d)^2\cos^2 v-(r\pm d)r_{uu})-r_u^2\sin^2 v}{((r\pm d)^2\cos^2 v+r_u^2)^2}$$

and

$$\tilde{H} = -\frac{\cos v(3r_u^2 + 2(r \pm d)^2 \cos^2 v - (r \pm d) r_{uu})}{2((r \pm d)^2 \cos^2 v + r_u^2)^{3/2}}$$

ii) If the radius function $r(u,v)$ be a v-parameter function then the Gaussian and mean curvature of F_d

$$\tilde{K} = \frac{(r_v \sin v + (r \pm d)\cos v)(2r_v^2 + (r \pm d)^2 - (r \pm d) r_{vv})}{(r \pm d)\cos v((r \pm d)^2 + r_v^2)^2}$$

and

$$\tilde{H} = -\frac{(r_v \sin v + (r \pm d)\cos v)((r \pm d)^2 + r_v^2) + (r \pm d)\cos v(2r_v^2 + (r \pm d)^2 - (r \pm d) r_{vv})}{2(r \pm d)\cos v((r \pm d)^2 + r_v^2)^{3/2}}$$

Corollary 11. Let F_d be a conchoidal surface of F given with the parametrization (5.2). If u-parameter radius function

$$r(u) = \pm d + \left(\frac{(2c^2 - 1)(-c_1 c_2 \sin(2\sqrt{2c^2 - 1}\,u) + \sin^2(\sqrt{2c^2 - 1}\,u)(c_1^2 - c_2^2) + c_2^2}{c^4} \right)^{\frac{-c^2}{2(2c^2 - 1)}}$$

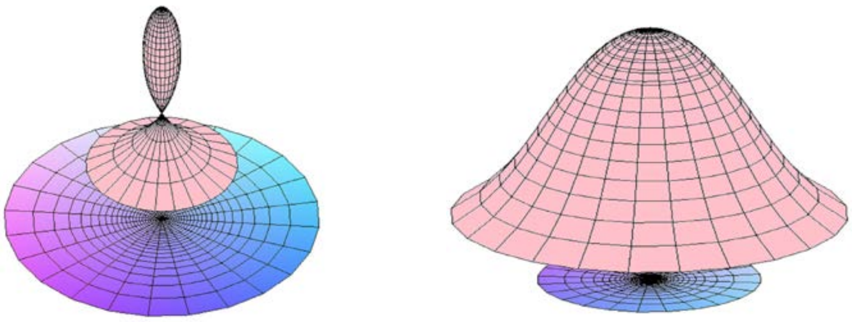
then F_d is flat and if

$$r(u) = \pm \frac{\sqrt{c}}{\sqrt{c_3 \sin(2cu) - c_4 \cos(2cu)}} \pm d$$

then F_d is minimal where $c = \cos v$ and c_1, c_2, c_3, c_4 are real constants.

Corollary 12. Let F_d be a conchoidal surface of F given with the parametrization (5.2). If v-parameter radius function $r(v) = \mp d + \dfrac{1}{c_1 \sin v - c_2 \cos v}$ then F_d is flat and also if $c_2 = 0$ then F_d is minimal.

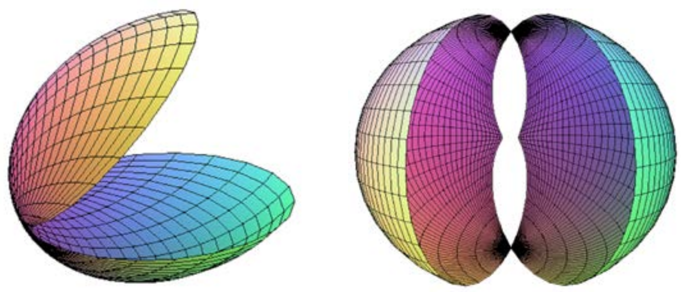
Example 3. 1) Let F be a plane then $f(u,v) = \dfrac{1}{\sin v}(\cos u \cos v, \sin u \cos v, \sin v)$ and its conchoidal surface $f_d(u,v) = (\dfrac{1}{\sin v} \pm d)(\cos u \cos v, \sin u \cos v, \sin v)$, (Figure 5a,b).



a) $d=-2$ b) $d=3$

Figure 5. Plane and its conchoidal surface.

2) Let the radius function $r(u,v)=\sin u\cos v$ then $f(u,v)=(\sin u\cos u\cos^2 v,\sin^2 u\cos^2 v,\sin u\cos v\sin v)$ which is a surface like a seashell (Figure 6a) and its conchoidal surface $f_d(u,v)=(\sin u\cos v\pm d)(\cos u\cos v,\sin u\cos v,\sin v)$, (Figure 6b).



a) b)

Figure 6. Seashell and its conchoidal.

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